# PERFECTLY MEAGER SETS AND UNIVERSALLY NULL SETS

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ABSTRACT. We will show that there is no ZFC example of a set distinguishing between universally null and perfectly meager sets.

## 1. INTRODUCTION

Consider the following three families of sets of reals:

**Definition 1.** *Let $X \subseteq \mathbb{R}$.*

1. *$X$ is perfectly meager if for every perfect set $P \subseteq \mathbb{R}$, $P \cap X$ is meager in $P$.*
2. *$X$ is universally meager if every Borel isomorphic image of $X$ is meager,*
3. *$X$ is universal null if every Borel isomorphic image of $X$ has Lebesgue measure zero.*

*Let* **PM**, **UM** *and* **UN** *denote these families respectively.*

One gets an equivalent definition of **UN** by replacing "Borel isomorphic" by "homeomorphic", but this is not the case with **UM**.

Let $\mathcal{M}$ and $\mathcal{N}$ denote the $\sigma$-ideals of meager and of measure zero subsets of the reals, respectively.

For a $\sigma$-ideal $\mathcal{J} \subseteq P(\mathbb{R})$ let

$$\mathsf{non}(\mathcal{J}) = \min\{|X| : X \subseteq \mathbb{R} \ \&\ X \not\subseteq \mathcal{J}\}.$$

There are many ZFC examples of uncountable sets that are in $\mathbf{UM} \cap \mathbf{UN}$. These include $\omega_1\omega_1^\star$-gaps, a selector from the constituents of a non-Borel $\mathbf{\Pi}^1_1$ set, etc. (see [9]) All these sets have size $\aleph_1$, since Miller [8] showed that, consistently, no set of size $2^{\aleph_0}$ is in $\mathbf{UM} \cup \mathbf{UN}$.

Grzegorek found other constructions in ZFC that produce sets of (consistently) different sizes.

**Theorem 2** (Grzegorek, [6])**.** 1. *There exists a set $X \in$ **UN** such that $|X| = \mathsf{non}(\mathcal{N})$,*
2. *There exists a set $X \in$ **UM** such that $|X| = \mathsf{non}(\mathcal{M})$.*

The problem whether the equality **UM** = **UN** is consistent is open. However, both inclusions are consistent with ZFC; $\mathbf{UM} \subseteq \mathbf{UN}$ holds in a model obtained by adding $\aleph_2$ Cohen reals, and $\mathbf{UN} \subseteq \mathbf{UM}$ holds in a model obtained by adding $\aleph_2$ random reals (side-by-side) (see [9], [8]).

In this paper we investigate the connection between families **UN** and **PM**, and show that both inclusions $\mathbf{PM} \subseteq \mathbf{UN}$ and $\mathbf{UN} \subseteq \mathbf{PM}$ are consistent with ZFC

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as well. Observe that trivially $\mathbf{UM} \subseteq \mathbf{PM}$, thus we only need to check that $\mathbf{PM} \subseteq \mathbf{UN}$ is consistent. Recall that $\mathbf{PM} \neq \mathbf{UM}$ is consistent ([12]) as well as $\mathbf{PM} = \mathbf{UM}$ ([2]).

We will show that:

**Theorem 3.** *It is consistent with* ZFC *that*

$$\mathbf{PM} \subseteq [\mathbb{R}]^{\leq \aleph_1} \subseteq \mathbf{UN}.$$

## 2. Forcing

Suppose that $X \subseteq 2^\omega$ is a perfectly meager set in $\mathbf{V}$. Let $\widetilde{P}$ be a fixed closed subset of $2^\omega \times 2^\omega$ which is universal for perfect sets in $2^\omega$. In other words, for ever perfect set $P \subseteq 2^\omega$ there exists $x$ such that $P = (\widetilde{P})_x$. Since $X$ is perfectly meager, we can find a sets $\widetilde{Q^n} \subseteq 2^\omega \times 2^\omega$ such that for every $x \in 2^\omega$, and $n \in \omega$,

1. $(\widetilde{Q^n})_x$ is a closed nowhere dense subset of $(\widetilde{P})_x$,
2. $X \cap (\widetilde{P})_x \subseteq (\bigcup_{n\in\omega} \widetilde{Q^n})_x$.

Clearly, the set $\bigcup_{n\in\omega} \widetilde{Q^n}$ witnesses that $X \in \mathbf{PM}$ since

$$X \subseteq 2^\omega \setminus \bigcup_{x\in 2^\omega} (\widetilde{P} \setminus \bigcup_{n\in\omega} \widetilde{Q^n})_x.$$

Note that the last inclusion makes sense even if $X$ is not a subset of $\mathbf{V}$. Suppose that $\mathbf{V}' \subseteq \mathbf{V}$ and $X \subseteq \mathbf{V}$ is a set of reals. We will say $\mathbf{V}' \models X \in \mathbf{PM}$ if there exists a family $\{\widetilde{Q^n} : n \in \omega\} \in \mathbf{V}'$ such that $X \cap (\widetilde{P})_x \subseteq (\bigcup_{n\in\omega} \widetilde{Q^n})_x$ for every real $x \in \mathbf{V}'$.

The property of being perfectly meager is not absolute so whether $X$ is perfectly meager in $\mathbf{V}'$ has no bearing onto whether $X$ is perfectly meager in $\mathbf{V}$. For example, if $x \in \mathbf{V}$ is a Cohen real over $\mathbf{V}'$ then the set $\{x\}$ is perfectly meager in $\mathbf{V}$ but not in $\mathbf{V}'$.

**Lemma 4.** *Let $\langle \mathcal{P}_\alpha, \dot{\mathcal{Q}}_\alpha : \alpha < \omega_2\}$ be a countable support iteration of proper forcing notions over $\mathbf{V} \models$ CH. Suppose that $X \subseteq \mathbf{V}^{\mathcal{P}_{\omega_2}} \cap \mathbb{R}$ is a perfectly meager set. Then there exists an $\omega_1$-club $C \subseteq \omega_2$ such that for every $\alpha \in C$,*

$$\mathbf{V}^{\mathcal{P}_\alpha} \models X \in \mathbf{PM}.$$

*Proof.* Let $\{\widetilde{Q^n} : n \in \omega\} \in \mathbf{V}^{\mathcal{P}_{\omega_2}}$ be a family witnessing that $X$ is perfectly meager. Let $C$ consist of those ordinals of cofinality $\omega_1$ that for every $n$, $\widetilde{Q^n} \cap \big((2^\omega \cap \mathbf{V}^{\mathcal{P}_\alpha}) \times 2^\omega\big) \in \mathbf{V}^{\mathcal{P}_\alpha}$. The usual argument involving Skolem-Löwenheim theorem shows that $C$ has the required property. $\square$

Our objective is to find a set of general conditions on a forcing notion $\mathbb{P}$ such that the countable support iteration of $\mathbb{P}$ of length $\omega_2$ produces a model where $\mathbf{PM} \subseteq [\mathbb{R}]^{\leq\aleph_1} \subseteq \mathbf{UN}$. These conditions are sufficient for the class of forcing notions defined using norms [10].

These conditions are the following:

1. $\mathbf{V}^{\mathbb{P}} \models \mathbf{V} \cap 2^\omega \in \mathcal{N}$,
2. $\mathbf{V}^{\mathbb{P}} \models \mathbf{V} \cap 2^\omega \notin \mathcal{M}$,
3. $\mathbb{P}$ is $\omega^\omega$-bounding, that is $\omega^\omega \cap \mathbf{V}$ is a dominating family in $\omega^\omega \cap \mathbf{V}^{\mathbb{P}}$,
4. $\mathbb{P}$ adds a real $x_{\mathbb{P}} \in 2^\omega$ such that $\mathbf{V} \models \{x_{\mathbb{P}}\} \notin \mathbf{PM}$.
5. $\mathbb{P}$ generic real is minimal, that is, if $g$ is $\mathbb{P}$-generic over $\mathbf{V}$ and $x \in \mathbf{V}[g] \cap 2^\omega$ then $x \in \mathbf{V}$ or $g \in \mathbf{V}[x]$.

Condition (1) is necessary to make all sets of size $\aleph_1$ universally null, and condition (2) is necessary to avoid making all $\aleph_1$ sets perfectly meager. Recall that (2) and (3) together are essentially equivalent to

$$\mathbf{V}^{\mathbb{P}} \models \mathbf{V} \cap \mathcal{M} \text{ is cofinal in } \mathcal{M}.$$

For the forcing notions $\mathbb{P}$ that we have in mind the following property holds: for every real $x \in \mathbf{V}^{\mathbb{P}}$ there exists a continuous function $f \in \mathbf{V}$ such that $x = f(x_G)$, where $x_G$ is a generic real.

Condition (5), guarantees that in the above context $f$ can be chosen to be a homeomorphism. In particular, if $X$ is a set of reals of size $\aleph_2$ then $X$ will contain a homeomorphic image of a sequence of generic reals.

The following forcing notion appeared in [5], it is similar (but not identical) to the infinitely equal real forcing from [7].

For a tree $p$ and $t \in p$, let $\mathsf{succ}_p(t)$ be the set of all immediate successors of $t$ in $p$, $p_t = \{v \in p : t \subseteq v \text{ or } v \subseteq t\}$ the subtree of $p$ determined by $t$, $p{\restriction}n$ the $n$-th level of $p$, and let $[p]$ be the set of branches of $p$. By identifying $s \in \omega^{<\omega}$ with the full-branching tree having root $s$, we can also denote $[s] = \{f \in \omega^\omega : s \subseteq f\}$.

Fix a strictly increasing function $f \in \omega^\omega$ and let $\mathbf{X} = \prod_{n\in\omega} f(n)$. Note that $\mathbf{X}$ is a Polish space homeomorphic to $2^\omega$. For technical reasons we require that $f(n) = 2^{\tilde{f}(n)}$ for $n \in \omega$.

Let $\mathbb{EE}$ be the following forcing notion:

$$p \in \mathbb{EE} \iff p \subseteq \bigcup_{n\in\omega} \bigcup_{j<n} \prod f(j) \text{ is a perfect tree } \&$$
$$\forall s \in p\ \exists t \in p\ \Big(s \subseteq t \ \&\ \mathsf{succ}_p(t) = f(|p|)\Big).$$

For $p, q \in \mathbb{EE}$, $p \geq q$ if $p \subseteq q$. Without loss of generality we can assume that $|\mathsf{succ}_p(s)| = 1$ or $\mathsf{succ}_p(s) = f(|p|)$ for all $p \in \mathbb{EE}$ and $s \in p$. Conditions of this type form a dense subset of $\mathbb{EE}$.

Let

$$\mathsf{split}(p) = \{s \in p : |\mathsf{succ}_p(s)| > 1\} = \bigcup_{n\in\omega} \mathsf{split}_n(p),$$

where $\mathsf{split}_n(p) = \Big\{s \in \mathsf{split}(p) : \Big|\big\{t \subsetneq s : t \in \mathsf{split}(p)\big\}\Big| = n\Big\}$.

For $p, q \in \mathbb{EE}$, $n \in \omega$, we let

$$p \geq_n q \iff p \geq q \ \&\ \mathsf{split}_n(q) = \mathsf{split}_n(p).$$

**Lemma 5** ([5]). 1. $\mathbb{EE}$ *satisfies Axiom A, so it is proper,*

2. $\mathbf{V}^{\mathbb{EE}} \models \mathbf{V} \cap 2^\omega \in \mathcal{N}$,
3. $\mathbf{V}^{\mathbb{EE}} \models \mathbf{V} \cap 2^\omega \notin \mathcal{M}$,
4. *for every maximal antichain $\mathcal{A} \subseteq \mathbb{EE}$, $p \in \mathbb{EE}$, and $n \in \omega$ there exists $q \geq_n p$ such that $\{r \in \mathcal{A} : r$ is compatible with $q\}$ is finite.*
5. *for every family of maximal antichains $\{\mathcal{A}_n : n \in \omega\}$ and $p \in \mathbb{EE}$ there exists $q \geq p$ such that for every $n$, $\{r \in \mathcal{A}_n : r$ is compatible with $q\}$ is finite.*
6. $\mathbb{EE}$ *is $\omega^\omega$ bounding,*
7. $\mathbf{V}^{\mathbb{EE}} \models \mathbf{V} \cap \mathcal{M}$ *is cofinal in $\mathcal{M}$.* □

Note that for $p \in \mathbb{EE}$ the set $[p]$ is a compact subset of $\mathbf{X} = \prod_n f(n)$. Moreover, there is a canonical isomorphism between $[p]$ and $2^\omega$ defined as follows:

For every $n$ let $\{s_0^n, \dots, s_{f(n)}^n\}$ be a fixed enumeration of 0-1 sequences of length $\tilde{f}(n)$ (recall that $f(n) = 2^{\tilde{f}(n)}$). Define $F : [p] \longrightarrow 2^\omega$ as

$$F(x) = s^{n_0}_{x(n_0+1)} {}^\frown s^{n_1}_{x(n_1+1)} {}^\frown \dots,$$

where $n_0, n_1, \dots$ is the increasing enumeration of the set $\{n : x{\restriction}n \in \mathsf{split}(p)\}$.

**Lemma 6.** *Let $p \in \mathbb{EE}$ and suppose that $H \subseteq [p]$ is a meager set in $[p]$. For every $n \in \omega$ there exists $q \geq_n p$ such that $[q] \cap H = \emptyset$. In particular,*

$$\Vdash_{\mathbb{EE}} \text{“}\mathbf{V} \models \{\dot{g}\} \notin \mathbf{PM}\text{”}.$$

*Proof.* Let $H \subseteq [p]$ be a meager set, and let $n \in \omega$. Fix a descending sequence of open sets $\langle U_k : k \in \omega\rangle$ such that each $U_k$ is dense in $[p]$ and $H \cap \bigcap_k U_k = \emptyset$. By induction build a sequence $\langle p_k : k \in \omega\rangle$ such that $p_0 = p$, and for every $k$,

1. $p_{k+1} \geq_{n+k+1} p_k \in \mathbb{EE}$,
2. $[p_{k+1}] \subseteq U_k$.

Suppose that $p_k$ is given. For every $v \in \mathsf{split}_{n+k+1}(p_k)$ find $q_v \geq (p_k)_v$ such that $[q_v] \subseteq U_k$. Let $p_{k+1} = \bigcup\{q_v : v \in \mathsf{split}_{n+k+1}(p_k)\}$. Condition $q = \lim_k p_k$ has the required property.

Suppose that $\{\widetilde{Q^n} : n \in \omega\} \in \mathbf{V}$ is a possible witness that $\{\dot{g}\}$ is perfectly meager, and let $p \in \mathbb{EE}$. Find $x \in \mathbf{V}$ such that $[p] = (P)_x$ and let $q \geq p$ be such that $[q] \cap \left(\bigcup_n \widetilde{Q^n}\right)_x = \emptyset$. Clearly,

$$q \Vdash_{\mathbb{EE}} \{\dot{g}\} \in \bigcup_{x \in \mathbf{V}} \left(P \setminus \bigcup_n \widetilde{Q^n}\right)_x.$$

In particular,

$$q \Vdash_{\mathbb{EE}} \text{“}\mathbf{V} \models \{\dot{g}\} \notin \mathbf{PM}\text{”}.$$

$\square$

**Lemma 7.** *Suppose that $p \in \mathbb{EE}$ and $p \Vdash_{\mathbb{EE}} \dot{x} \in 2^\omega$. For every $n \in \omega$ there exists $q \geq_n p$ and a continuous function $F : [q] \longrightarrow 2^\omega$ such that*

$$q \Vdash_{\mathbb{EE}} \dot{x} = F(\dot{g}),$$

*where $\dot{g}$ is the canonical name for the generic real.*

*Moreover, we can require that for every $v \in \mathsf{split}_n(q)$ and any $x_1, x_2 \in [q_v]$, $F(x_1){\restriction}n = F(x_2){\restriction}n$.*

*Proof.* The first part is a special case of a more general fact. For $n \in \omega$ let $\mathcal{A}_n \subseteq \mathbb{EE}$ be a maximal antichain below $p$ such that

$$\forall r \in \mathcal{A}_n \ \exists s \in 2^n \ r \Vdash_{\mathbb{EE}} \dot{x}{\restriction}n = s.$$

Use 5(5) to find $q \geq p$ such that for every $n \in \omega$,

$$\{r \in \mathcal{A}_n : r \text{ is compatible with } q\}$$

is finite. Let $\mathcal{A}'_n = \{r \in \mathcal{A}_n : r \text{ is compatible with } q\}$. Without loss of generality we can assume that $[q] \subseteq \bigcup_{r \in \mathcal{A}'_n}[r]$. It follows that $[r] \cap [q]$ is clopen in $[q]$ for every $r \in \mathcal{A}'_n$. Define $F : [q] \longrightarrow 2^\omega$ as $F(x) = y$ if for every $n \in \omega$ there exists $r \in \mathcal{A}'_n$ such that $x \in [r]$ and $r \Vdash_{\mathbb{EE}} \dot{x}{\restriction}n = y{\restriction}n$. It is easy to see that $F$ is a continuous function that has the required properties.

To show the second part we need to build $q$ in such a way that for every $v \in \mathsf{split}_n(q)$, there is $r \in \mathcal{A}'_n$ such that $q_v \geq r$. $\square$

**Lemma 8.** *Suppose that $p \in \mathbb{EE}$, $n \in \omega$ and $p \Vdash_{\mathbb{EE}} \dot{x} \in 2^\omega$. Let $F : [q] \longrightarrow 2^\omega$ be a continuous function such that $p \Vdash_{\mathbb{EE}} \dot{x} = F(\dot{g})$.*

*There exists $q \geq p$ such that $F{\restriction}[q]$ is constant, or there exists $q \geq_n p$ such that $F{\restriction}[q]$ is one-to-one. In particular, the generic real is minimal.*

*Proof.* Consider the following two cases.

CASE 1 $p \not\Vdash_{\mathbb{EE}} \dot{x} \notin \mathbf{V}$. Let $x \in \mathbf{V}$ and $q \geq p$ be such that $q \Vdash_{\mathbb{EE}} \dot{x} = x$. Clearly $F{\restriction}[q]$ is constant with value $x$.

CASE 2 $p \Vdash_{\mathbb{EE}} \dot{x} \notin \mathbf{V}$.

Build by induction a sequence of conditions $\langle p_k : k \in \omega \rangle$ such that $p_0 = p$ and for every $k$,

1. $p_{k+1} \geq_{n+k+1} p_k$,
2. sets $\left\{F\text{"}\Big(\big[(p_{k+1})_s\big]\Big) : s \in \mathsf{split}_{n+k+1}(p_{k+1})\right\}$ are pairwise disjoint and have diameter $< 2^{-k}$

Suppose that $p_k$ is given. Note that $F\text{"}\big([(p_k)_s]\big)$ is uncountable for every $s \in p_k$. For $v \in \mathsf{split}_{n+k+1}(p_k)$ choose pairwise different reals $x_v \in F\text{"}\big([(p_k)_v]\big)$. It is not important now but will be relevant in the sequel, that we can choose these reals "effectively" from a fixed countable subset of $[p_k]$. Let $\ell > k$ be such that sequences $x_v{\restriction}\ell$ are also pairwise different. For every $v \in \mathsf{split}_{n+k+1}(p_k)$ let $s_v \in \mathsf{split}(p_k)$ be such that for every $z \in [(p_k)_{s_v}]$, $F(z){\restriction}\ell = x_v{\restriction}\ell$. If $F$ is as in the second part of lemma 7 then we can find $s_v$ in $\mathsf{split}_\ell(p_k)$. Define

$$p_{k+1} = \bigcup \{(p_k)_{s_v} : v \in \mathsf{split}_{n+k+1}(p_k)\}.$$

Observe that $q = \lim_k p_k$ has the required property. $\square$

Note that the above lemma shows that the reals added by $\mathbb{EE}$ are minimal. Infinitely equal forcing from [7] or [4] does not have this property.

## 3. ITERATION OF $\mathbb{EE}$.

Let $\alpha \leq \omega_2$ be an ordinal and suppose that $\mathbb{EE}_\alpha$ is a countable support iteration of $\mathbb{EE}$ of length $\alpha$. In other words, $p \in \mathbb{EE}_\alpha$ is

1. $p$ is a function and $\mathsf{dom}(p) = \alpha$,
2. $\mathsf{supp}(p) = \{\beta : p(\beta) \neq \emptyset\}$ is countable,
3. $\forall \beta < \alpha \; p{\restriction}\beta \Vdash_{\mathbb{EE}_\beta} p(\beta) \in \mathbb{EE}$.

For $F \in [\alpha]^{<\omega}$, $n \in \omega$, and $p, q \in \mathbb{EE}_\alpha$ define

$$q \geq_{F,n} p \iff q \geq p \;\&\; \forall \beta \in F \; q{\restriction}\beta \Vdash_{\mathbb{EE}_\beta} q(\beta) \geq_n p(\beta).$$

The following fact is well-known.

**Theorem 9** ([5], [7], [3])**.** *Suppose that $p \in \mathbb{EE}_\alpha$, $F \in [\alpha]^{<\omega}$, and $n \in \omega$.*

1. *for every maximal antichain $\mathcal{A} \subseteq \mathbb{EE}_\alpha$, there exists $q \geq_{F,n} p$ such that $\{r \in \mathcal{A} : r$ is compatible with $q\}$ is finite.*
2. *for every family of maximal antichains $\{\mathcal{A}_n : n \in \omega\}$ there exists $q \geq p$ such that for every $n$, $\{r \in \mathcal{A}_n : r$ is compatible with $q\}$ is finite.*
3. $\mathbf{V}^{\mathbb{EE}_{\omega_2}} \models [\mathbb{R}]^{<2^{\aleph_0}} \subseteq \mathcal{N}$.
4. $\mathbf{V}^{\mathbb{EE}_{\omega_2}} \models \mathcal{M} \cap \mathbf{V}$ *is cofinal in* $\mathcal{M}$. $\square$

For $p \in \mathbb{EE}_\alpha$ let $\mathsf{cl}(p)$ be the smallest set $w \subseteq \alpha$ such that $p$ can be evaluated using generic reals $\langle \dot{g}_\beta : \beta \in w\rangle$. In other words, $\mathsf{cl}(p)$ consists of those $\beta < \alpha$ such that the transitive closure of $p$ contains $\mathbb{EE}_\beta$ name for an element of $\mathbb{EE}$. It is well-known [11] that $\{p \in \mathbb{EE}_\alpha : \mathsf{cl}(p) \in [\alpha]^{\leq\omega}\}$ is dense in $\mathbb{EE}_\alpha$.

Suppose that $p \in \mathbb{EE}_\alpha$, $w = \mathsf{cl}(p)$ is countable and $\alpha_p = \mathsf{ot}\big(\mathsf{cl}(p)\big)$. Let $\mathbb{EE}_w$ be the countable support iteration of $\mathbb{EE}$ with the domain $w$. In other words, consider The countable support iteration $\langle \mathcal{P}_\beta, \dot{\mathcal{Q}}_\beta : \beta < \sup(w)\rangle$ such that

$$\forall \beta < \sup(w) \ \Vdash_{\mathcal{P}_\beta} \dot{\mathcal{Q}}_\beta \simeq \begin{cases} \mathbb{EE} & \text{if } \beta \in w \\ \emptyset & \text{if } \beta \notin w \end{cases}.$$

It is clear that $\mathbb{EE}_w \simeq \mathbb{EE}_{\alpha_p}$. Moreover, we can view condition $p$ as a member of $\mathbb{EE}_w$.

For the rest of the section we will consider only the iteration of $\mathbb{EE}$ of countable length $\alpha$ and show that $\mathbb{EE}_\alpha$ has the same properties that $\mathbb{EE}$.

Let $\alpha$ be a countable ordinal and $p \in \mathbb{EE}_\alpha$. Define $\overline{p} \subseteq \mathbf{X}^\alpha$ as follows:

$\langle x_\beta : \beta < \alpha\rangle \in \overline{p}$ if for every $\beta < \alpha$,

$$x_\beta \in \Big[p(\beta)\big[\langle x_\gamma : \gamma < \beta\rangle\big]\Big].$$

Note that $p(\beta)[\langle x_\gamma : \gamma < \beta\rangle]$ is the interpretation of $p(\beta)$ using reals $\langle x_\gamma : \gamma < \beta\rangle$ so may be undefined if these reals are not sufficiently generic.

For a set $G \subseteq \mathbf{X}^\alpha$, $u \subseteq \alpha$, and $x \in \mathbf{X}^u$ let

$$(G)_x = \{y \in \mathbf{X}^{\alpha\setminus u} : \exists z \in G \ z{\restriction}u = x \ \& \ z{\restriction}(\alpha \setminus u) = y\},$$

and for $\beta \in \alpha$ let

$$(G)_\beta = \{x(\beta) : x \in G\}.$$

We say that $p \in \mathbb{EE}_\alpha$ is good if

1. $\overline{p}$ is compact,
2. for every $\beta < \alpha$ and $x \in \overline{p{\restriction}\beta}$, $\overline{p[x]} = (p)_x$ and $\overline{p(\beta)[x]} = ((p)_x)_\beta$.
3. $\overline{p}$ is homeomorphic to $\mathbf{X}^\alpha$ via a homeomorphism $h$ such that for every $\beta < \alpha$ and $x \in \overline{p{\restriction}\beta}$, $h{\restriction}((p)_x)_\beta$ is a homeomorphism between $((p)_x)_\beta$ and $\mathbf{X}$.

**Lemma 10.** *$\{p \in \mathbb{EE}_\alpha : \overline{p}$ is good$\}$ is dense in $\mathbb{EE}_\alpha$.*

*Proof.* Case 1. $\alpha = \beta + 1$.

Fix $p \in \mathbb{EE}_\alpha$ and for $n \in \omega$ let $\mathcal{A}_n$ be a maximal antichain below $p{\restriction}\beta$ such that

1. $\forall r \in \mathcal{A}_n \ \overline{r}$ is compact.
2. $\forall r \in \mathcal{A}_n \ \exists t \subseteq \prod_{j<n} f(j) \ r \Vdash_{\mathbb{EE}_\beta} p(\beta){\restriction}n = t$.

Fix a sequence $\langle F_n : n \in \omega\rangle$ such that for $n \in \omega$,

1. $F_n \in [\beta]^{<\omega}$,
2. $F_n \subseteq F_{n+1}$,
3. $\bigcup_n F_n = \beta$.

By induction build a sequence $\langle q_n : n \in \omega\rangle$ such that for $n \in \omega$,

1. $\overline{q_n}$ is compact,
2. $q_{n+1} \geq_{F_n,n} q_n$,
3. $\exists \mathcal{A}'_n \in [\mathcal{A}_n]^{<\omega} \ \overline{q_n} \subseteq \bigcup_{r\in\mathcal{A}'_n} \overline{r}$.

Let $q_\omega = \lim_n q_n$. As in the proof of 7 we show that there exists a continuous function $F : \overline{q_\omega} \longrightarrow \mathbb{EE}$ (encode elements of $\mathbb{EE}$ as reals) such that

$$q_\omega \Vdash_{\mathbb{EE}_\beta} p(\beta) = F(\langle \dot{g}_\gamma : \gamma < \beta\rangle).$$

Consider $q = q_\omega {}^\frown p(\beta) \geq p$. Clearly,

$$\overline{q} = \{\langle x, y\rangle : x \in \overline{q_\omega},\ y \in [F(x)]\}$$

is compact in $\mathbf{X}^\alpha$. Remaining requirements are met as well.

CASE 2. $\alpha$ is limit.

Given $p \in \mathbb{EE}_\alpha$ fix sequences $\langle F_n : n \in \omega\rangle$ and $\langle \alpha_n : n \in \omega\rangle$ such that

1. $F_n \in [\alpha_n]^{<\omega}$,
2. $F_n \subseteq F_{n+1}$,
3. $\bigcup_n F_n = \alpha$,
4. $\sup_n \alpha_n = \alpha$.

By induction build a sequence $\langle q_n : n \in \omega\rangle$ such that for $n \in \omega$,

1. $q_n \in \mathbb{EE}_\alpha$,
2. $\mathsf{supp}(q_n) \subseteq \alpha_n$,
3. $q_{n+1} \geq_{F_n,n} q_n$,
4. $q_n{\restriction}\alpha_n \geq p{\restriction}\alpha_n$,
5. $\overline{q_n{\restriction}\alpha_n}$ is compact in $\mathbf{X}^{\alpha_n}$.

Let $q = \lim_n q_n$. Note that $\overline{q} = \bigcap_n \overline{q_n{\restriction}\alpha_n} \times \mathbf{X}^{\alpha\setminus\alpha_n}$ is as required. □

From now on we will always work with conditions $p$ such that $\overline{p}$ is good. We noticed earlier that for every condition $p \in \mathbb{EE}$, $[p]$ is canonically isomorphic to $2^\omega$, in exactly the same way we can verify that if $p \in \mathbb{EE}_\alpha$ and $\overline{p}$ is good then $\overline{p}$ is isomorphic to $(2^\omega)^\alpha$.

As in the lemma 7 we show that:

**Lemma 11.** *Suppose that $p \in \mathbb{EE}_\alpha$ and $p \Vdash_{\mathbb{EE}_\alpha} \dot{x} \in 2^\omega$. Then there exists $q \geq p$ and a continuous function $F : \overline{p} \longrightarrow 2^\omega$ such that*

$$q \Vdash_{\mathbb{EE}_\alpha} \dot{x} = F(\dot{\mathbf{g}}),$$

*where $\dot{\mathbf{g}} = \langle \dot{g}_\beta : \beta < \alpha\rangle$ is the sequence of generic reals.*

**Lemma 12.** *Let $p \in \mathbb{EE}_\alpha$ and suppose that $H \subseteq \overline{p}$ is a meager set in $\overline{p}$. For every $F \in [\alpha]^{<\omega}$ and $n \in \omega$ there exists $q \geq_{F,n} p$ such that $\overline{q} \cap H = \emptyset$.*

*Proof.* As before, without loss of generality we can assume that $\alpha$ is countable.

Induction on $\alpha$.

CASE 1. $\alpha = \beta + 1$.

Suppose that $p \in \mathbb{EE}_\alpha$ and $H \subseteq \overline{p} \subseteq \mathbf{X}^\beta \times \mathbf{X}$ is meager, and let $F \in [\alpha]^{<\omega}$ and $n \in \omega$ be given.

Let

$$H' = \left\{x \in \overline{p{\restriction}\beta} : (H)_x \text{ is not meager in } \left[p(\beta)[x]\right] = ((\overline{p})_x)_\beta\right\}.$$

Using the fact that $\overline{p}$ is homeomorphic to $(2^\omega)^\alpha$ via homeomorphism respecting vertical sections, and by Kuratowski-Ulam theorem, we conclude that $H'$ is a meager set in $\overline{p{\restriction}\beta}$.

Recall the following classical lemma:

**Lemma 13** ([1])**.** *Suppose that $H \subseteq 2^\omega \times 2^\omega$ is a Borel set.*

1. *Assume $(H)_x$ is meager for all $x$. Then there exists a sequence of Borel sets $\{G_n : n \in \omega\} \subseteq 2^\omega \times 2^\omega$ such that*
   (a) *$(G_n)_x$ is a closed nowhere dense set for all $x \in 2^\omega$,*
   (b) *$H \subseteq \bigcup_{n\in\omega} G_n$.*

By the inductive hypothesis we can find $q^\star \geq_{F\cap\beta,n} p{\restriction}\beta$ such that $\overline{q^\star} \cap H' = \emptyset$. By lemma 6 for every $x \in \overline{q^\star}$ there exists $q_x \geq_n p(\beta)[x]$ such that $[q_x] \cap (H)_x = \emptyset$. Moreover, by 13, the mapping $x \mapsto q_x$ is can be chosen to be Borel, and subsequently, by shrinking $q^\star$, continuous. Let $q \in \mathbb{EE}_\alpha$ be defined such that $q{\restriction}\beta = q^\star$ and $q^\star \Vdash_{\mathbb{EE}_\beta} q(\beta) = q_{\dot{g}_\beta}$. It is clear that $q$ has the required properties.

Case 2. $\alpha$ is limit.

Fix sequences $\langle F_n : n \in \omega\rangle$ and $\langle \alpha_n : n \in \omega\rangle$ such that

1. $F_n \in [\alpha_n]^{<\omega}$,
2. $F_n \subseteq F_{n+1}$,
3. $\bigcup_n F_n = \alpha$,
4. $\sup_n \alpha_n = \alpha$.

By induction build a sequence $\langle q_n : n \in \omega\rangle$ such that for $n \in \omega$,

1. $q_n \in \mathbb{EE}_\alpha$,
2. $\mathsf{supp}(q_n) \subseteq \alpha_n$,
3. $q_{n+1} \geq_{F_n,n} q_n$,
4. $q_n{\restriction}\alpha_n \geq p{\restriction}\alpha_n$,
5. $\overline{q_n{\restriction}\alpha_n} \cap H_n = \emptyset$, where $H_n = \left\{x \in \overline{q_n{\restriction}\alpha_n} : (H)_x \text{ is not meager in } \overline{p[x]}\right\}$.

As before (5) is possible by Kuratowski-Ulam theorem. Let $q = \lim_n q_n$. It is clear that $\overline{q} \cap H = \emptyset$. □

The following lemma is an analog of lemma 8.

**Lemma 14.** *Suppose that $p \in \mathbb{EE}_\alpha$, $n \in \omega$ and $p \Vdash_{\mathbb{EE}_\alpha} \dot{x} \in 2^\omega$. Let $F : \overline{p} \longrightarrow 2^\omega$ be a continuous function such that $p \Vdash_{\mathbb{EE}_\alpha} \dot{x} = F(\dot{\mathbf{g}})$, where $\dot{\mathbf{g}} = \langle \dot{g}_\beta : \beta < \alpha\rangle$ is the sequence of generic reals. There exists $q \geq p$ such that exactly one of the following conditions hold:*

1. *$F{\restriction}\overline{q}$ is constant,*
2. *there exists $\beta < \alpha$ such that $F{\restriction}\overline{q{\restriction}\beta}$ is one-to-one and for every $x \in \overline{q{\restriction}\beta}$, $F{\restriction}(\overline{q{\restriction}\beta})_x$ is constant,*
3. *$F{\restriction}\overline{q}$ is one-to-one.*

*Proof.* We have three cases:

Case 1. There exists $q \geq p$ such that $q \Vdash_{\mathbb{EE}_\alpha} \dot{x} \in \mathbf{V}$. Without loss of generality we can assume that for some $x \in \mathbf{V} \cap 2^\omega$ $q \Vdash_{\mathbb{EE}_\alpha} \dot{x} = x$. It follows that $F{\restriction}\overline{q}$ is constant.

Case 2. There exists $q \geq p$ such that $q \Vdash_{\mathbb{EE}_\alpha} \exists\beta < \alpha\ \dot{x} \in \mathbf{V}^{\mathbb{EE}_\beta}$. By shrinking $q$ we can assume that there exists a continuous function $G : \overline{q{\restriction}\beta} \longrightarrow 2^\omega$ such that $q \Vdash_{\mathbb{EE}_\alpha} \dot{x} = G(\dot{\mathbf{g}}{\restriction}\beta)$. In particular, for $x \in [q]$, $F(x) = G(x{\restriction}\beta)$. If $\beta$ was minimal then, using the argument below, we can also assume that $G$ is one-to-one.

Suppose that $q \in \mathbb{EE}_\alpha$, $F \in [\alpha]^{<\omega}$, and $n \in \omega$. Without loss of generality we can assume that for every $\beta \in F$, $q{\restriction}\beta$ determines the value of $\mathsf{split}_n(q(\beta))$ (up to finitely many values). Suppose that $\sigma : F \longrightarrow \omega^{<\omega}$ is a function such that $\sigma(\beta) \in \mathsf{split}_n(q(\beta))$ for $\beta \in F$. Let $(q)_\sigma$ be the condition defined as

$$\forall\beta < \alpha\ (q)_\sigma{\restriction}\beta \Vdash_{\mathbb{EE}_\beta} (q)_\sigma(\beta) = \begin{cases} q(\beta) & \text{if } \beta \notin F \\ (q(\beta))_{\sigma(\beta)} & \text{if } \beta \in F \end{cases}.$$

Let $\Sigma_{F,n}$ be the finite set of all mappings $\sigma$ satisfying the requirements.

**Lemma 15.** *Suppose that $F \in [\alpha]^{<\omega}$, $n \in \omega$ and*

$$p \Vdash_{\mathbb{EE}_\alpha} \dot{x} = F(\dot{\mathbf{g}}) \ \& \ \forall \beta < \alpha \ \dot{x} \notin \mathbf{V}^{\mathbb{EE}_\beta}.$$

*There exists $q \geq_{F,n} p$ such that the sets $\left\{F\text{"}\left(\overline{(q)_\sigma}\right) : \sigma \in \Sigma_{F,n}\right\}$ are pairwise disjoint.*

*Proof.* Induction on $|F|$ and $\alpha$. If $F = \{\beta\}$ this is essentially lemma 8.

Let $\{v_j : j < k^\star\}$ be an enumeration of $\mathsf{split}_n(p(\beta))$. For $v \in \mathsf{split}_n(p)$ choose pairwise different reals $x_v \in F\text{"}\left(\overline{(p)_v}\right)$. Note that this choice can be made canonically from, for example, the countable dense let of leftmost branches of subtrees of $p$. Let $\ell > k$ be such that sequences $x_v \restriction \ell$ are also pairwise different. Define conditions $\langle r_j : j \leq k^\star\rangle$, $\langle q_j : j \leq k^\star\rangle$ such that for every $j \leq k^\star$,

1. $r_j \in \mathbb{EE}_\beta$,
2. $r_{j+1} \geq r_j$,
3. $r_j \Vdash_{\mathbb{EE}_\beta} q_j \geq (p)_{v_j} \restriction [\beta, \alpha)$,
4. $\forall z \in \overline{r_j {}^\frown q_j}$, $F(z)\restriction \ell = F(x_{v_j})\restriction \ell$.

Let $q\restriction\beta = q_{k^\star}$ and $q\restriction[\beta,\alpha) = \bigcup_{j<k^\star} q_j$.

Suppose that $|F| = k+1$ and let $\beta = \max(F)$.

By the part already proved, for each $\mathbf{x} = \langle x_\gamma : \gamma < \beta\rangle \in \overline{p\restriction\beta}$ find a condition $q_{\mathbf{x}} \geq_n p\restriction[\beta,\alpha)[\mathbf{x}]$ such that the sets $\left\{F\text{"}\left(\overline{(q_{\mathbf{x}})_s}\right) : s \in \mathsf{split}_n(q_{\mathbf{x}})\right\}$ are pairwise disjoint. Note that we can do it in such a way that the mapping $\mathbf{x} \mapsto q_{\mathbf{x}}$ is continuous (As before we first choose $q_{\mathbf{x}}$ in a Borel way, and then shrink $p\restriction\beta$ to make this mapping continuous). That defines a $\mathbb{EE}_\beta$-name for an element of $\mathbb{EE}_{\beta,\alpha}$, which we call $q^\star$.

Next, let $F' = F \setminus \{\beta\}$ and apply the inductive hypothesis to find $q' \geq_{F',n} p\restriction\beta$ such that $\left\{F\text{"}\left(\overline{(q')_\sigma}\right) : \sigma \in \Sigma_{F',n}\right\}$ are pairwise disjoint. Let $q \in \mathbb{EE}_\alpha$ be defined as $q\restriction\beta = q'$ and $q\restriction\beta \Vdash_{\mathbb{EE}_\beta} q\restriction[\beta,\alpha) = q^\star$.

It is clear that $q$ is as required. □

Case 3. $p \Vdash_{\mathbb{EE}_\alpha} \forall\beta < \alpha \ \dot{x} \notin \mathbf{V}^{\mathbb{EE}_\beta}$.

Let $\langle F_n : k \in \omega\rangle$ be an increasing sequence of finite sets such that $\bigcup_n F_n = \alpha$.

By induction build a sequence of conditions $\langle p_n : n \in \omega\rangle$ such that $p_0 = p$ and for every $n$,

1. $p_{n+1} \geq_{F_n,n} p_n$,
2. sets $\left\{F\text{"}\left(\overline{(p_n)_\sigma}\right) : \sigma \in \Sigma_{F_n,n}\right\}$ are pairwise disjoint.

Let $q = \lim_n p_n$.

Suppose that $\mathbf{x} = \langle x_\beta : \beta < \alpha\rangle$ and $\mathbf{x}' = \langle x'_\beta : \beta < \alpha\rangle$ are two distinct points in $\overline{q}$. Let $\beta$ be the first ordinal such that $x_\beta \neq x'_\beta$. Let $n$ be so large that $\beta \in F_n$ and there are two distinct $\sigma, \sigma' \in \Sigma_{F_n,n}$ such that $\mathbf{x} \in \overline{(p_n)_\sigma}$ and $\mathbf{x}' \in \overline{(p_n)_{\sigma'}}$. Since $F\text{"}\left(\overline{(p_n)_\sigma}\right) \cap F\text{"}\left(\overline{(p_n)_{\sigma'}}\right) = \emptyset$, it follows that $F(\mathbf{x}) \neq F(\mathbf{x}')$. □

## 4. A model where $\mathbf{PM} \subseteq \mathbf{UN}$.

Let $\mathbb{EE}_{\omega_2}$ be the countable support iteration of $\mathbb{EE}$ of length $\aleph_2$. We will show that in $\mathbf{V}^{\mathbb{EE}_{\omega_2}}$, $\mathbf{PM} \subseteq \mathbf{UN}$.

By theorem 9(2), $\mathbf{V}^{\mathbb{EE}_{\omega_2}} \models [\mathbb{R}]^{<2^{\aleph_0}} \subseteq \mathbf{UN}$, thus we have to show that

$$\mathbf{V}^{\mathbb{EE}_{\omega_2}} \models \mathbf{PM} \subseteq [\mathbb{R}]^{<2^{\aleph_0}}.$$

Suppose that $X \in \mathbf{V}^{\mathbb{EE}_{\omega_2}}$ is a set of reals of size $\aleph_2$. Let $\{\dot{x}_\alpha : \alpha < \omega_2\}$ be the set of names for elements of $X$ such that $\Vdash_{\mathbb{EE}_{\omega_2}} \forall \alpha \neq \beta \; \dot{x}_\alpha \neq \dot{x}_\beta$. Apply lemma 11 and find for each $\alpha < \omega_2$ a set $w_\alpha \in [\omega_2]^{\leq \omega}$, a condition $p_\alpha \in \mathbb{EE}_{w_\alpha}$, and a continuous function $F_\alpha : \overline{p_\alpha} \longrightarrow 2^\omega$ such that

$$p_\alpha \Vdash_{\mathbb{EE}_{\omega_2}} \dot{x}_\alpha = F_\alpha(\langle \dot{g}_\beta : \beta \in w_\alpha \rangle).$$

We can assume that $w_\alpha$ is minimal. In other words,

$$p_\alpha \Vdash_{\mathbb{EE}_{\omega_2}} \forall \beta < \sup(w_\alpha) \; \dot{x}_\alpha \notin \mathbf{V}^{\mathbb{EE}_\beta}.$$

In particular, without loss of generality we can assume $F_\alpha$ is one-to-one, so it is a homeomorphism.

By thinning out we can assume that $\mathsf{ot}(w_\alpha) = \gamma$, $F_\alpha = F$ and $\overline{p_\alpha} = \overline{p}$. Moreover, since $\mathbf{V} \models \mathsf{CH}$, we can assume that $w_\alpha \cap w_\beta = w^\star$ for $\alpha \neq \beta$. Finally, without loss of generality we can assume that $w^\star = \emptyset$.

Let $P = F"(\overline{p})$. Since $F$ is a homeomorphism, $P$ is perfect. We will show that $X \cap P$ is not meager in $\mathbf{V}^{\mathbb{EE}_{\omega_2}}$ (relative to $P$).

Assume otherwise and let $H \subseteq P$ be a meager set such that for some $p^\star \in \mathbb{EE}_{\omega_2}$, $p^\star \Vdash_{\mathbb{EE}_{\omega_2}} X \cap P \subseteq H$. By 9(4) we can assume that $H \in \mathbf{V}$. Set $G = (F)^{-1}(H)$ and notice that $G$ is a meager subset of $\overline{p}$.

Find $\alpha < \omega_2$ such that $w_\alpha \cap \mathsf{cl}(p^\star) = \emptyset$. By lemma 12 there exists $q \geq p$, $q \in \mathbb{EE}_{w_\alpha} \simeq \mathbb{EE}_\gamma$ such that $\overline{q} \cap G = \emptyset$.

Since $p^\star$ and $q$ are compatible let $r \geq p^\star, q$. It follows that

$$r \Vdash_{\mathbb{EE}_{\omega_2}} \dot{x}_\alpha = F_\alpha(\langle \dot{q}_\beta : \beta \in w_\alpha \rangle) \notin H,$$

which finishes the proof.

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